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Orthosymmetries and Jordan Triples

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Orthosymmetric ortholattices (OSOLs) have been introduced in order to approximate Hilbertian lattices (ortholattices of closed subspaces of a Hilbert space). Axioms of OSOLs are selected properties of usual orthogonal symmetries on a Hilbertian lattice and many posets defined by means of associative or Jordan algebras possess a set of automorphisms satisfying these axioms. In this paper, we describe and illustrate a method using Jordan triples and which provides a common setting for the study of orthosymmetries in associative algebras and Jordan algebras.

INTRODUCTION

Orthosymmetric ortholattices (abbreviated OSOLs) have been introduced in order to approximate Hilbertian lattices (orthomodular lattices of closed subspaces of a Hilbert space) more closely than by orthomodular lattices (Mayet, 1992). Any Hilbertian lattice carries a natural (Mayet, 1992) and unique structure of OSOL, and the main result of Chevalier (1995a) asserts that, more generally, any orthomodular lattice of projections of a Rickart *-ring, satisfying $2x = 0 \Rightarrow x = 0$, also possesses a natural structure of OSOL. There exist similar examples in Jordan algebras and the orthomodular lattice of all idempotents of a JBW-algebra is an OSOL. This result is announced in Chevalier (1995a) and proved in Chevalier (1995b). The aim of this paper is to find a common setting for this kind of result, and it seems that this setting is provided by Jordan triples. M. Edwards and G. Rüttimann have shown in many papers the interest of such structures in functional analysis and they have also studied their properties related to orthomodular structures (Edwards and Rüttiman, 1995).

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The paper is organized as follows. In Section 1, we recall the definition of a weak generalized orthomodular poset (abbreviated WGOMP) and define orthosymmetric WGOMPs. Orthosymmetric WGOMPs can be embedded in orthosymmetric orthomodular posets in the same way as WGOMPs are embedded in orthomodular posets. Section 2 is devoted to Jordan triples: definition of the WGOMPs $\mathcal{U}(A)$ of all tripotents of a Jordan triple A, characterization of commutativity in the poset $\mathcal{U}(A)$, and study of automorphisms of $\mathcal{U}(A)$. The difficulty of a definition of a natural structure of orthosymmetric WGOMP on $\mathcal{U}(A)$ in the general case is noticed. In the final section, examples show how previous results apply to associative algebras with involution and JB-algebras.

For notions concerning orthomodular structures and the logicoalgebraic approach to quantum mechanics, let we refer to Pták and Pulmannová (1991). For Jordan operator algebras Hanche-Olsen and Størmer (1984) is a standard reference, and for information about triple products see the papers by M. Edwards and G. Rüttimann, for example those quoted in Edwards and Rüttimann (1995).

1. WGOMPs AND ORTHOSYMMETRIC WGOMPs

A WGOMP is a poset with a least element 0 such that every interval $[0, a]$ is an orthomodular poset. More precisely:

Definition 1. (Mayet-Ippolito, 1991). Let $(A, \leq)$ be a poset with a least element 0, such that every interval $[0, a]$ is equipped with a unary operation $x \to x^{\perp_a}$. A is a weak generalized orthomodular poset (WGOMP) if it satisfies the following conditions:

C1. For all $a \in A$, $([0, a], \leq, \perp_a)$ is an orthomodular poset.
C2. If $a \leq b \leq c$, then $a^{\perp_b} = b \wedge a^{\perp_c}$.

Elements a and b are said to be orthogonal (in notation $a \perp b$) if $a, b \leq c$ and $a \leq b^{\perp_c}$ for some $c \in A$.

C3. If $a \perp b$, then $a \vee b$ exists.
C4. If $a \perp b$, $c \perp a$, and $c \perp b$, then $c \perp a \vee b$.

A generalized orthomodular poset (abbreviated GOMP) A is a WGOMP satisfying the supplementary condition:

C5. If $c \perp a$ and $c \perp b$ and if $a \vee b$ exists, then $c \perp a \vee b$.

Examples of WGOMPs and GOMPs

1. Orthomodular lattices, orthomodular posets, and generalized orthomodular lattices in the sense of M. Janowitz are GOMPs.

2. Any *-ring with a proper involution ($xx^* = 0 \Rightarrow x = 0$) and, in particular, any C^*-algebra is a GOMP (Mayet-Ippolito, 1991; Hedlíková and Pulmannová, 1995). Definitions of the order and orthogonality relations are as follows:

$$a \leq b \Leftrightarrow a^*a = a^*b \quad \text{and} \quad aa^* = ba^* \qquad a^{\perp_b} = b - a \quad \text{if} \quad a \leq b$$

3. Any JB-algebra with order and orthogonality relations defined by

$$a \leq b \Leftrightarrow a^2b = a^3, \qquad a^{\perp_b} = b - a \quad \text{if} \quad a \leq b$$

is a GOMP (Chevalier, 1993; Hedliková and Pulmannová, 1995).

Any WGOMP A can be embedded as an order ideal in an orthomodular poset $\hat{A}$ (Mayet-Ippolito, 1991) so that, for any $x \in \hat{A}$, $x \in A$ or $x^\perp \in A$. If A is a GOMP, then the embedding preserves all existing suprema of two elements.

Two elements a and b of a WGOMP A commute (in notation: $a \leftrightarrow b$) if there exist three pairwise orthogonal elements of A, a_1, b_1, and c, such that $a = c \vee a_1$ and $b = c \vee b_1$. Two elements a and b of a WGOMP A commute if and only if they commute in the orthomodular poset $\hat{A}$.

Orthosymmetric ortholattices has been introduced in Mayet (1992) and their definition has been improved and generalized to orthoposets in Mayet and Pulmannová (1995). A generalization to WGOMP is also possible.

Definition 2. A WGOMP A is said to be orthosymmetric if, for all $a \in A$, there exists a mapping S_a: $A \to A$, called an orthosymmetry, such that:

O1. $S_a^2 = Id_A$;
O2. If $x \leq b$, then $S_a(x^{\perp_b}) = S_a(x)^{\perp_{S_a(b)}}$;
O3. $x \leq y \Leftrightarrow S_a(x) \leq S_a(y)$;
O4. $a \perp b \Rightarrow S_a \circ S_b = S_{a \vee b}$;
O5. $S_a \circ S_b \circ S_a = S_{S_a(b)}$ ($\Leftrightarrow S_a \circ S_b = S_{S_a(b)} \circ S_a$);
O6. $S_a(x) = x \Leftrightarrow x \leftrightarrow a$.

If A is an orthoposet or an orthomodular lattice, this definition is not new: an orthosymmetric orthoposet is a orthosymmetric orthoposet (OSOP) in the sense of Mayet and Pulmannová (1995), an orthosymmetric orthomodular lattice is an orthosymmetric ortholattice (OSOL) in the sense of Mayet (1992).

Proposition 1. Any orthosymmetric WGOMP can be embedded as an order ideal in an OSOP.

Proof. Let A be an orthosymmetric WGOMP and $\hat{A}$ be the orthomodular poset in which A can be embedded as an order ideal. If $a \in A$ and $x \in \hat{A} \backslash A$, we extend S_a to x by $S_a(x) = S_a(x^\perp)^\perp$ and if now $a \in \hat{A} \backslash A$, we extend S to

a by $S_a = S_{a^\perp}$. Elementary computations show that $\hat{A}$, endowed with the family of mappings $(S_a)_{a \in \hat{A}}$, is an OSOP.

2. JORDAN TRIPLES

Most of the results of this section related to the triple product are true in a more general setting, for example, in Jordan triple systems as defined in McCrimmon (1982). In order to avoid complexity, we consider in this paper that a Jordan triple A is a real or complex vector space equipped with a triple product symmetric-bilinear in the outer variables, conjugate linear in the middle variable (in the complex case), and which satisfies, for all elements a, b, c, d, and e,

$$\{ab\{cde\}\} - \{cd\{abe\}\} = \{\{abc\}de\} - \{c\{bad\}e\}$$

For a pair of elements a and b in A, let $D(a, b)$ and $Q(a, b)$ be the mappings from A^2 to A defined by

$$D(a, b)x = \{abx\}, \qquad Q(a, b)x = \{axb\}$$

$D(a, a)$ and $Q(a, a)$ are denoted by $D(a)$ and $Q(a)$. An element a in a Jordan triple is said to be a tripotent if $\{aaa\} = a$. For each tripotent a the Peirce projections are the mutually orthogonal projections $P_0(a)$, $P_1(a)$, and $P_2(a)$ defined by

$$P_0(a) = id_A - 2D(a) + Q^2(a),$$

$$P_1(a) = 2(D(a) - Q^2(a)),$$

$$P_2(a) = Q^2(a)$$

Their ranges $A_0(a)$, $A_1(a)$, $A_2(a)$ satisfy the following relations:

$$A = A_0(a) \oplus A_1(a) \oplus A_2(a)$$

$$\{A_2(a)A_0(a)A\} = \{A_0(a)A_2(a)A\} = \{0\}$$

$$\begin{aligned} \{A_i(a)A_j(a)A_k(a)\} &\subseteq A_{i-j+k}(a) \quad \text{if} \quad 0 \leq i - j + k \leq 2 \\ &= \{0\} \qquad \text{otherwise} \end{aligned}$$

The Jordan triple A is said to be anisotropic if $\{xxx\} = 0$ implies $x = 0$. In this case, the binary relation $D(a, b) = 0$ is symmetric and if $D(a, a) = 0$, then $a = 0$. This relation is called the orthogonality relation on A and is denoted by $\perp$. The set $\mathcal{U}(A)$ of all tripotents of a Jordan triple A can be ordered by the relation $a \leq b$ if $b - a$ is a tripotent orthogonal to a.

For each tripotent a, the mapping $\perp_a$: $b \in [0, a] \rightarrow b^{\perp^a} = \mathrm{a} - \mathrm{b} \in [0, a]$ is an orthocomplementation on the poset $[0, a]$ which is an orthomodular poset and $(\mathcal{U}(A), \leq, \{\perp_a | a \in \mathcal{U}(A)\})$ is a WGOMP (Edwards and Rüttimann, 1995). In the search for orthosymmetries defined on $\mathcal{U}(A)$ the importance of Pierce projections leads us to determine linear combinations of these mappings which satisfy properties similar to axioms of OSOLs.

More precisely, let λ_0, λ_1, and λ_2 be nonzero scalars. For any tripotent a define a linear mapping S_a by $S_a = \lambda_0 P_0(a) + \lambda_1 P_1(a) + \lambda_2 P_2(a)$. There exist μ_0, μ_1, μ_2 such that $S_a = \mu_0 id_A + \mu_1 D(a) + \mu_2 P_2(a)$.

Proposition 2. 1. If, for $i - j + k \in \{0, 1, 2\}$, $\lambda_i \overline{\lambda_j} \lambda_k = \lambda_{i-j+k}$, then S_a is a triple automorphism and $S_a^{-1} = \lambda_0^{-1}\, P_0(a) + \lambda_1^{-1}\, P_1(a) + \lambda_2^{-1}\, P_2(a)$. Conversely, if S_a is an automorphism and if, for $i - j + k \in \{0, 1, 2\}$, $\{A_i(a) A_j(a) A_k(a)\} \neq \{0\}$, then $i - j + k \in \{0, 1, 2\}$ imply $\lambda_i \overline{\lambda_j} \lambda_k = \lambda_{i-j+k}$.

2. If S_a and S_b are triple automorphisms, then $S_a \circ S_b = S_{S_a(b)} \circ S_a$.

3. If a and b are orthogonal tripotents and if S_a and S_b are triple automorphisms, then $\mu_1^2 = 4\mu_2$ and $\mu_0 = 1$ imply $S_a \circ S_b = S_b \circ S_a = S_{a \vee b}$.

4. Involutive triple automorphisms satisfying $S_a \circ S_b = S_{S_a(b)} \circ S_a$ and $S_a \circ S_b = S_b \circ S_a = S_{a \vee b}$ if $a \perp b$ are given by $S_a = P_0(a) + tP_1(a) + t^2 P_2(a)$, $t \in \{-1, +1\}$.

Proof. (1) For the proof of $S_a(\{xyz\}) = \{S_a(x) S_a(y) S_a(z)\}$ decompose x, y, and z by means of $A = A_0(a) \oplus A_1(a) \oplus A_2(a)$ and use $\{A_2(a)\ A_0(a) A\} = \{A_0(a) A_2(a) A\} = \{0\}$, $\{A_i(a) A_j(a) A_k(a)\} \subseteq A_{i-j+k}(a)$ if $0 \leq i - j + k \leq 2$ and is $\{0\}$ otherwise.

(2) The proof is an easy calculation.

(3) Apply the following lemma, the proof of which uses standard identities satisfied by Jordan triples (see McCrimmon, 1982).

Lemma 1. If a and b are two orthogonal tripotents in a Jordan triple A, then:

1. $Q(a)Q(b) = D(a)Q(b) = Q(b)D(a) = Q(a)Q(a, b) = Q(a, b)Q(a) = 0$.

2. $D(a + b) = D(a) + D(b)$ and $4Q^2(a, b) = Q(a + b)^2 - Q(a)^2 - Q(b)^2 = 4D(a)D(b)$.

(4) It is an obvious consequence of (1)–(3).

Clearly, involutive triple automorphisms preserve order and orthogonality relations and thus Part 4 of the proposition shows that $S_a = P_0(a) - P_1(a) + P_2(a)$ is the unique linear combination of Peirce projections, different from the identity, satisfying five of the six axioms of orthosymmetric WGOMPs. In the sequel, S_a always denotes this mapping, which is called the Peirce reflection defined by the tripotent a. Notice that $S_a = Id_A - 2P_1(a)$ and so S_a is the usual symmetry associated to the projection $P_1(a)$ of the vector space A. Axiom O6 of orthosymmetric WGOMPs means that the set of fixed points of S_a is the commutant of $\{a\}$ and, in general, this axiom fails to hold (see Example 1 below).

A study of the two relations $S_a(b) = b$ and $a \leftrightarrow b$ is necessary. It is easy to prove that, for $a, b \in \mathcal{U}(A)$, $S_a(b) = b$ is equivalent to $\{aab\} = \{a\{aba\}a\}$ or $b \in A_0(a) \oplus A_2(a)$ and the following proposition characterizes tripotents which commute in the lattice meaning:

Proposition 3. Two tripotents a and b Jordan *-triple A commute if and only if

$$\{aab\} = \{aba\} = \{bba\} = \{bab\}$$

Proof. If a and b commute, then there exist three pairwise orthogonal tripotents a_1, b_1, and c such that $a = c \vee a_1$ and $b = c \vee b_1$. By using this decomposition of a and b, the proof of $\{aab\} = \{aba\} = \{bba\} = \{bab\}$ is easy. Conversely if $\{aab\} = \{aba\} = \{bba\} = \{bab\}$, then $c = \{aab\}$ is a tripotent and $c \leq a$, $c \leq b$. Define two tripotents $a_1 = a - c$ and $b_1 = b - c$. We have $a = c \vee a_1$ and $b = c \vee b_1$ with a_1, b_1, and c pairwise orthogonal tripotents.

One can prove that $a \leq b$ is equivalent to $a = \{aab\} = \{aba\} = \{bba\} = \{bab\}$ and $a \perp b$ if and only if $0 = \{aab\} = \{aba\} = \{bba\} = \{bab\}$.

3. EXAMPLES

Example 1. Let A be a unital JB-algebra with an anisotropic triple product defined by $\{abc\} = a(bc) - b(ac) + (ab)c$. Tripotents are elements satisfying $a^3 = a$ and any idempotent is a tripotent. For two tripotents a and b, $S_a(b) = b - 8a^2b + 8a^2(a^2b)$ and $S_a(b) = b$ is equivalent to $a^2b = a^2(a^2b)$. Since A is a unital JB-algebra, this is also equivalent to a^2 and b operators commute. In the WGOMP of all tripotents of A, $a^2 \leftrightarrow a$ if and only if $a^2 = a$ and thus, since a^2 operator commutes with a, $S_a(b) = b$ is not equivalent to $a \leftrightarrow b$. Now consider the set $Idem(A) = [0, 1]$ of all idempotents of A. For a and b in $Idem(A)$, $S_a(b) = b - 8ab + 8a(ab) = U_{2a-1}(b)$, $S_a(b) \in Idem(A)$, and $S_a(b) = b$ if and only if $a \leftrightarrow b$. Thus, all the axioms of orthosymmetric WGOMPs hold true in the poset $Idem(A)$ equipped with the set of automorphisms $(S_a)_{a \in Idem A}$. In the particular case of a JBW-algebra (a JB-algebra which is also a dual Banach space) $Idem(A)$ is a lattice and thus an orthosymmetric ortholattice.

Example 2. Consider a *-algebra A with a proper involution and the triple product $\{abc\} = 1/2(ab^*c + cb^*a)$. It is an anisotropic Jordan triple in which tripotents are partial isometries. Any projection is a tripotent and, for a projection p, the restriction of the Peirce reflection S_p to the WGOMP

$Proj(A)$ of all projections of A is defined by $S_p(q) = (2p - 1)q(2p - 1)$. For two projections p and q, $\{ppq\} = p$ is equivalent to $pq = p$ and so the order defined by means of the triple product is the usual order on a set of projections. We have $S_p(q) = q$ if and only if $pq = qp$, which is equivalent to $p \leftrightarrow q$ and thus Axiom O6 is satisfied. The WGOMP $Proj(A)$, equipped with the family of automorphisms $(S_p)_{p \in Proj(A)}$, is orthosymmetric and, in particular, the set of all projections of a C^*-algebra is an orthosymmetric WGOMP.

If A is also a Rickart *-ring, $Proj(A)$ is an orthomodular lattice and we have obtained a proof of the main result of Chevalier (1995a) by using Jordan triples.

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REFERENCES

Chevalier, G. (1993). Order and orthogonality relations in rings and algebras, *Tatra Mountains Mathematical Publications,* **3**, 31–46.

Chevalier, G. (1995a). Orthosymmetric ortholattices and Rickart *-rings, *International Journal of Theoretical Physics,* **34**, 1289–1296.

Chevalier, G. (1995b). An order relation in JB-algebras and JBW-algebras, *preprint.*

Edwards, C. M., and Rüttiman G. T. (1995). Lattice of tripotents in a JBW *-triple, *International Journal of Theoretical Physics,* **34**, 1349–1357.

Hanche-Olsen, H., and Størmer E. (1984). *Jordan Operators Algebras*, Pitman, London.

Mayet, R. (1992). Orthosymmetric ortholattices, *Proceedings of the American Mathematical Society,* **114**, 295–306.

Mayet, R., and Pulmannová, S. (1995). Nearly ortholattices and Hilbert spaces, *Foundations of Physics,* **24**, 1425–1437.

Mayet-Ippolito, A. (1991). Generalized orthomodular posets, *Demonstratio Mathematica,* **24**, 263–279.

McCrimmon, K. (1982). Compatible Peirce decompositions of Jordan triple systems, *Pacific Journal of Mathematics,* **103**, 57–102.

Hedliková, J., and Pulmannová, S. (1995). Generalized difference posets and orthoalgebras, *preprint.*

Pták P., and Pulmannová, S. (1991). *Orthomodular Structures as Quantum Logics*, Kluver, Dordrecht.